Involute Profile of Non-Circular Gears

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Abstract: This paper presents an approach to calculate the parameters of a family of noncircular gears. The calculations of the general elliptical rolling curves are based on complex algebraic methods. Two methods was developed to calculate the contour curves of involute profile. The first based on the virtual manufacturing in a CAD system, while the second by determines the contour curves analytically. These gears were manufactured by wire EDM technology.

Keywords: non-circular, involute, elliptical gears, Maple, EDM

1 Introduction

The earlier version of spatial non-circular gears shown on a sketch made by Leonardo da Vinchi. In the 17-18th century makes has more practical applications of the non-circular gears in the clockworks, musical instruments, mechanical theatres and another automatic toys.
There are some well-known non-circular gear models made by F. Reuleaux to study the kinematics in the technical education in the beginning years of 20th century. (Fig.1).

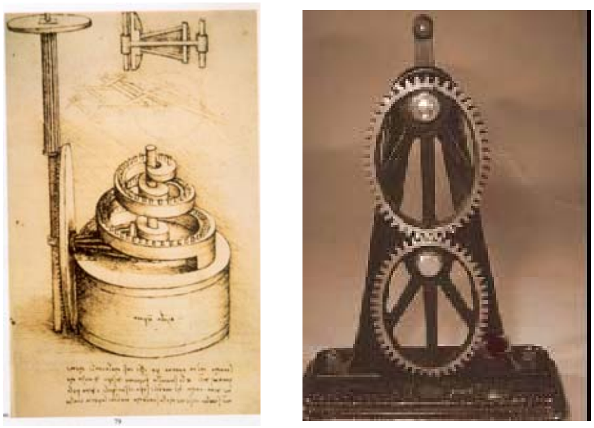


Figure 1

In the middle of 20th century the non-circular gearing was used in electromechanical systems to control and drive non-linear potentiometers. In spite of the improvement of digital techniques, the importance of non-circular gears are not decreased.
The profound summary and list of earlier references of this discipline is found in book [1]. For a huge lifework of Mr. F. L. Litvin [2] is a little-known part. In this work all basic questions of this special machinery elements are discussed exhaustively, but the applied calculation, construction and manufacturing methods are now anachronistic. In the corresponding chapters of new books [3-4] of the author summarise the most important theoretical results of [2].
The articles [5], [6] reflect some issues of the new constructions and manufacturing techniques. The [7] demonstrate the practical basics of the metrology of non-circular gears.
In this article a method is presented to calculate the basic parameters of general elliptical gears. The involute profile of teeth obtained by i.) virtual manufacturing in a CAD system and ii.) complex analytical method.
To formal derivation and numerical calculation we carried on in the symbolical mathematical program Maple® V. 7R.

2 General elliptical pitch curves

The polar equation of general elliptical curve is

$$\rho_k := \frac{p_k}{1 - e_k \cos(k\,\phi)} \quad p_k, e_k \in \mathbb{R} \quad k = 1,2,... \qquad (1)$$

where $\mathbb{R}$ is a set of real numbers[1].
The curve ρ_k hase k ρ_{kmax} maximal values and k ρ_{kmin} minimal values of its radius. The pitch curve have 2k congruent part between ρ_{kmax} and ρ_{kmin} values, that can be transformed to each other by rotation and/or mirroring. The involute profile of teeth's can be obtained by well-known generating (envelope) method.
The profiles of teeth curves located in the same positions of the congruent parts of pitch curve are identical, if the symmetric line of teeth of the basic rack coincide with the polar vector ρ_k corresponding $\varphi = 0$, and number of teeth is

$$z_k = 2k.(\,\xi + ½)\ (\xi = \ldots, 4, 5, \ldots) \qquad (2)$$

The number of teeth in one congruent part of the pitch curve is ξ + ½ (not integer!). If k1 = k2 the always the same teeth's of driven and driver wheel

[1] In case k = 1 the pitch curve is a classical ellipse. The rotation shaft coincides with one of its foci.

are connected, while in case of k1 ≠ k2 the teeth connecting with periodically identical position[2]. In The mean transmission ratio of gears is η_{mean} = k1/k2, and the ratio of actual relative angular velocities is $\eta = \rho_{k1}/\rho_{k2}$.
In practice to calculate parameters of pitch curves the origins k1, k2, η_{max} and the distance of axes „a" are given. The system of basic equations

$$\frac{p_{k1}\,(1-e_{k2})}{(1+e_{k1})\,p_{k2}} = A \qquad \frac{p_{k1}\,(1+e_{k2})}{(1-e_{k1})\,p_{k2}} = \eta_{max}$$

$$\frac{p_{k1}}{1-e_{k1}} + \frac{p_{k2}}{1+e_{k2}} = a \qquad \frac{p_{k1}}{1+e_{k1}} + \frac{p_{k2}}{1-e_{k2}} = a \tag{3}$$

is linearly depend on the formal parameter „A".
Solution of system (3) is

$$p_{k1} = \frac{2\,a\,A\,\eta_{max}}{A+\eta_{max}+2\,\eta_{max}\,A} \quad p_{k2} = \frac{2\,a}{A+\eta_{max}+2}$$
$$e_{k1} = \frac{-A+\eta_{max}}{A+\eta_{max}+2\,\eta_{max}\,A} \quad e_{k2} = \frac{-A+\eta_{max}}{A+\eta_{max}+2} \tag{4}$$

Substitute the solution (4) in to equation (1) can be the value of common modul of gears can be obtained (5)

$$m := \frac{\displaystyle\int_0^{\frac{\pi}{k}} \sqrt{\rho_k^{\,2} + \left(\frac{\partial}{\partial\phi}\rho_k\right)^2}\, d\phi}{\xi\,\pi} \tag{5}$$

where k = k1 or k = k2. (If the gears are manufactured with wire EDM technology, *m* can be an arbitrary value different from the standardised ones.)
The curvature radius of pitch curve is

$$K := \frac{\left(\rho^2 + \left(\frac{\partial}{\partial\phi}\rho\right)^2\right)^{\left(\frac{3}{2}\right)}}{\rho^2 + 2\left(\frac{\partial}{\partial\phi}\rho\right)^2 - \rho\left(\frac{\partial^2}{\partial\phi^2}\rho\right)} \tag{6}$$

In the practice usually K > 0 and the pitch curve is convex[3].

3 Virtual manufacturing in CAD system

The basic rack cutter consist of two straight lines that form a pressure angle α with respect to the real axes of complex co-ordinate system. The co-ordinates of corner points of first teeth in the basic rack are

$$Q_0 = m(h_1 - I(\frac{\pi}{4} + h_1.tan(\alpha))) \tag{7.a}$$
$$Q_1 = m(-h_2 - I(\frac{\pi}{4} - h_2.tan(\alpha))) \tag{7.b}$$
$$Q_2 = m(-h_2 + I(\frac{\pi}{4} - h_2.tan(\alpha))) \tag{7.c}$$
$$Q_3 = m(h_1 + I(\frac{\pi}{4} + h_2.tan(\alpha))) \tag{7.d}$$

where $I^2 = -1$, see Fig. 3. The pitch line of the cutter is coinciding with the complex axe. The fillet radiuses of standard tool are in this case neglected. Complex co-ordinates of sth corner point in the rack are

$$Q_s := Q_{s-4} + I.m.\pi \tag{8}$$

s = 4, 5, …, 4ζ – 1, where ζ = ξ + 1 is the total number of teeth of basic rack.
The pitch line of the cutter rolls on the pitch curve without slipping. Let the angular position of pitch curve (1) φ = ψ. The pitch line of the tool is actually the tangent of pitch curve (1). The complex co-ordinates of the sth corner point of the tool are:

$$w := \left(Q_s - I\int_0^{\psi}\sqrt{\rho^2 + \left(\frac{\partial}{\partial\phi}\rho\right)^2}\,d\phi\right)e^{(I\,\mu)} + \rho\,e^{(I\,\psi)} \tag{9}$$

and the slope angle of the tangent line is

$$\mu := \psi + \frac{\pi}{2} + \theta \tag{10}$$

The angle between radius vector and tangent line of (1) is

$$\theta := \arctan\left(\frac{\rho}{\frac{\partial}{\partial\phi}\rho}\right) \tag{11}$$

By evaluate the equations (1), (7-11) the complex co-ordinates of corner points of tool rack can be calculated. The first part of generated gear in case of *m* = 0.4973, *P* = 8.56996, *e* = 0.231692, *k* =2 shown on Fig. 2.

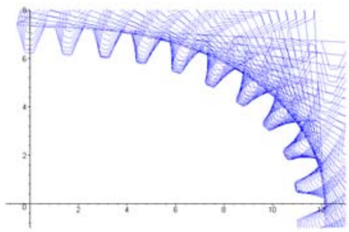


Figure 2

[2] This is in working practice unfavourable, but this is a basic property of non-circular gears.
[3] This is one advantage for generating technology with CAD, but with the method presented in chapter 4 the case K < 0 can be equally treated.

In the CAD systems standard manipulation commands like MOVE, ROTATE, SUBTRACT, and MIRROR can be used to virtually manufacture the gear. The movement of the tool object is determined by equations (9-11). In every step the tool object is subtracted from the workpiece object which results the final contour curve of the gear. Apply the appropriate mirroring and rotating functions the complete contour curve can be produced.

By manufacturing the gears with wire EDM cutting technology, the offset value *Δ* is very important an depend on the used technology, look Fig. 3.

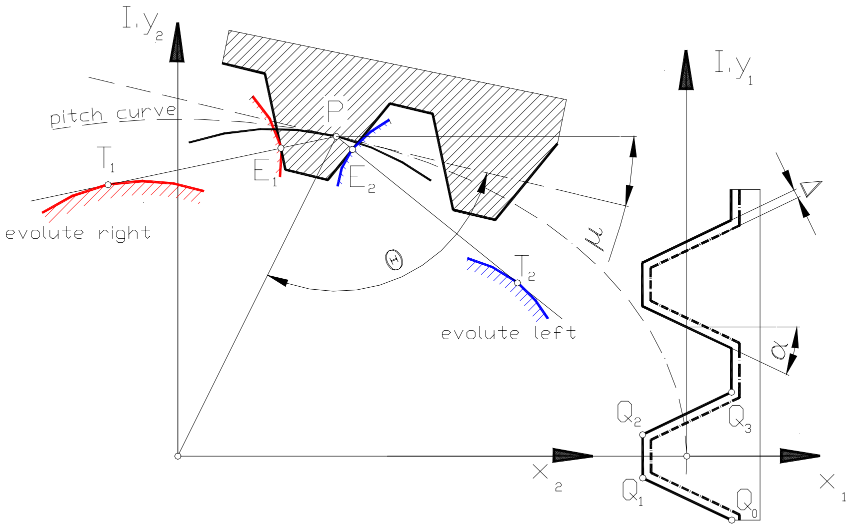


Figure 3

4 The basic curve and the involute profile

The basic involute profile curves of the teeth are determined by the enveloped figures of the profile normal of the teeth of the rack. The normal lines PT_1 and PT_2 coincide with the tangent line at the contact point P of the pitch curve (Willis-Kennedy Theorem).

The basic curves of the left and right profiles of the teeth are different. The complex equation of normal line parameterised by λ is

$$W = \rho.e^{i.\varphi} + \lambda.e^{i.(\varphi + \theta \pm \alpha)} \qquad (12)$$

The differential equation of the enveloped curve derived by

$$\tan(\mu) := \frac{\Im\left(\frac{\partial}{\partial\phi}W\right)}{\Re\left(\frac{\partial}{\partial\phi}W\right)} = \frac{\Im\left(\frac{\partial}{\partial\lambda}W\right)}{\Re\left(\frac{\partial}{\partial\lambda}W\right)} \qquad (13)$$

where I and R symbolise the real and imaginary part of partial derivatives of *W*. Solution of equation (13) is

$$\lambda := \frac{\left(\left(\frac{\partial}{\partial\phi}\rho\right)^2 + \rho^2\right)^{\left(\frac{3}{2}\right)} \sin(\alpha)}{2\left(\frac{\partial}{\partial\phi}\rho\right)^2 + \rho^2 - \rho(\phi)\left(\frac{\partial^2}{\partial\phi^2}\rho(\phi)\right)} \qquad (14)$$

Considering formula (6) it is evident that K = r.sin(α).

The continuous function of the rack profile curve is Q(u) see Fig, 4:

$$Q(u) = \begin{cases} \frac{a.u}{\beta}, u \le \beta \\ a, x \le \pi - \beta \\ \frac{a.(\pi - u)}{\beta}, u \le \chi \\ -b, x \le \chi \\ \frac{a.(u - 2\pi)}{\beta} & \text{otherwise} \end{cases} \qquad (15)$$

where

$$a = h_1 \quad b = h_2 \quad \beta = \frac{h_1}{tan(\alpha)} \quad \gamma = \pi + \frac{h_2}{tan(\alpha)} \qquad (16)$$

expanded in to odd type of trigonometric series

$$Q(u) = \frac{B_0}{2} + \sum_j A_{2j-1}\sin((2j-1)u) + B_{2j}\cos(2ju) \qquad (17)$$

The Fourier-coefficients are

$$A_k = \frac{1}{\pi}\int_0^{\pi} Q(u).\sin(k.u)du$$
$$B_k = \frac{1}{\pi}\int_0^{\pi} Q(u).\cos(k.u)du \qquad (18)$$

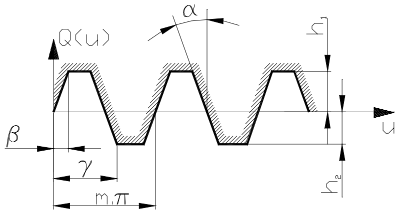


Figure 4.

Based on equation (9) the point of rack profiles curve at the angular position ψ and profile parameter u determined by equation (19)

$$w = (Q(u) - I\int_0^{\psi}\sqrt{\rho^2 + \left(\frac{\partial}{\partial\varphi}\rho\right)^2}\,d\varphi).e^{I\mu} + \rho.e^{I\psi} \qquad (19)$$

The involute points of teeth E_1 and E_2 are intersections of the tool curve (19) and the normal lines of teeth. The equation of normal lines is

$$k = \rho.e^{I\psi} + \Lambda.e^{I.(\mu \pm \frac{\pi}{2} \pm \alpha)} \qquad (20)$$

Solving the equations (19) and (20) Λ and u parameters can be calculated. Substitute Λ or u into equation (19) or (20) the co-ordinates of involute point can be determined.

5 The undercutting line of the teeth

In case of the conventional involute gears the teeth are undercutted if radius of pitch circle is smaller than

$$r_{\min} = \frac{m}{\sin(\alpha)^2} \qquad (21)$$

In case of non-circular gears, the teeth are undercutted if K < r_{min}. The approximate undercutting curve can be calculated by the sweeping path of corner point Q_{4s+1} and Q_{4s+2}. The limit point of undercutting is the intersection of exact involute profiles and the sweeping path curves of Q_{4s+1} and Q_{4s+2} points The part of undercutting line is shown in Figure 5.

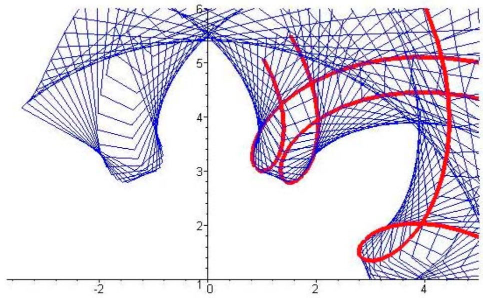
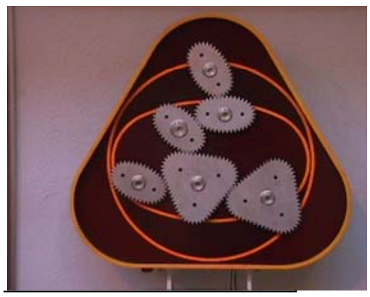


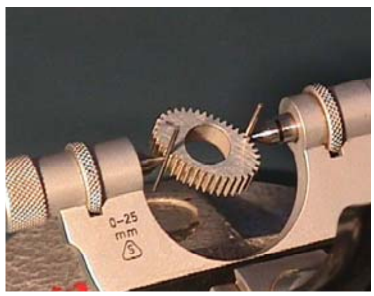
Figure 5

6 Applications

From the practice of author some products are shown in Fig. 6 which ware calculated with the presented method. Usually the gears in the industrial applications (flow measuring equipment's, printing machines and robots) have modul cca. 0.4 – 3.5 mm



Figures 6



7 Summary

This paper present an approach that calculates the parameters of a set of noncircular gears by general elliptical rolling curve applying complex algebraic methods. The complex formulas easy to use for calculating the undercutting limit. To formal derivation and numerical calculation ware carried in the symbolical mathematical program Maple® V. 7R. To manufacture the gears by wire EDM technology the final tool paths were generated in a CAD system.

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